INSTITUTO TECNOLÓGICO DE AERONÁUTICA

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EXACT SOLUTION FOR BUCKLING OF AXIALLY-COMPRESSED CYLINDRICAL PANELS WITH FRAMES ATTACHED TO THE CIRCULAR EDGES

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Orientador

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EXACT SOLUTION FOR BUCKLING OF AXIALLY-COMPRESSED CYLINDRICAL PANELS WITH FRAMES ATTACHED TO THE CIRCULAR EDGES

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Dedico este trabalho aos meus familiares, em especial ao meu pai, que nos deixou esse ano, mas estará sempre vivo em nossas lembranças.

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"A Civil 17 é uma grande família!".

Artur Silva Cantanhede

Resumo

Esse trabalho apresenta uma solução exata para o problema de valor de contorno que descreve a flambagem linear por compressão axial de painéis cilíndricos com reforçadores acoplados às bordas curvas. As condições de contorno diferem do problema clássico simplesmente apoiado, em geral assumido para fins de projeto, no sentido de que a torsão resistida pelos reforçados são levadas em conta. A presença dos reforçadores torna os resultados obtidos aqui de grande interesse prático e valiosos como dados para referência.

Abstract

This work presents an exact solution for the boundary-value problem which describes the linear buckling of axially-compressed cylindrical panels with frames attached to the circular edges. The boundary conditions differ from the classical simply supported ones, often assumed for design purposes, in the sense that the torsion resisted by the frames is also taken into account. The presence of the frames makes the results reported herein of practical interest and valuable as benchmark data.

List of Symbols

$(\ )_{,x}$	partial derivative with respect to x
$(\ )_{,y}$	partial derivative with respect to y
d/dx	Leibnitz notation for derivatives
$\partial/\partial x$	Another notation for partial derivatives of multivariable function
$(\ )', (\ )^{(m)}$	Lagrange's notation for derivatives
$\nabla^4(\ )$	two-dimensional harmonic operator
N_x, N_y	menbrane's normal forces
M_x, M_y	bending moments
$\bar{M}$	moment exerted by the frames on the circular edges
u, v, w	displacement components in the panel midsurface
ξ, η	adimensional coordinates
$\boldsymbol{K}$	Stiffness Matrix
W	work done by a force
$\boldsymbol{W}$	vector with unkown coefficients
E	Young's modulus
G	shear modulus
I	moment of inertia
J	torsional constant
Γ	warping constant
ν	Poisson's ratio
D	panel bending rigidity
a, b	panel dimensions

h	panel thickness
p_{cl}	minimum buckling load for a simply supported panel
δ	variational operator

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1 Introduction

Circular cylindrical panels are important structural components in many engineering applications, such as aircraft fuselages, exteriors of rockets, submarine hulls, offshore platforms, and many others. The skin of a fuselage, for instance, can be thought of as an assembly of cylindrical panels supported by frames (circumferential stiffeners) and stringers (longitudinal stiffeners). In typical design, these panels are the structural components to buckle first in order to exploit the structure load capacity efficiently [1,2]. An isolated panel, extracted from a skin portion between adjacent stiffeners, can thus be used as a simple solution domain to estimate the skin buckling.

The classical simply supported conditions, often assumed for design purposes, are represented by the requirement that the radial displacements and the bending moments be zero along the edges of an isolated panel, in addition to the constraint of null tangential displacements and normal in-plane stress resultants associated with the buckling deformation. Unlike the above-mentioned conditions, in this paper the bending moments along the panel circular edges are not made zero but equal to the torsion resisted by the attached frames. This will make these edges to be somewhere between simply supported and clamped. The linearized Donnell's equations, with prebuckling rotations neglected, are chosen to describe the panel buckling behavior [3].

The frames are supposed to resist torsion in two ways: the first is denoted by Saint-Venant torsion, the second by warping torsion [4]. For most practical cases, one contribution may be neglected as compared to the other. Warping torsion is usually negligible in bars with solid or thin-walled closed cross sections. Saint-Venant torsion, on the other hand, may be neglected in thin-walled open cross sections [5]. Our exact solution is independent of whether the Saint-Venant torsion or the warping torsion are resisted separately or together, and can be used to judge the relevance of each contribution if desired.

The exact solution for buckling of axially-compressed cylindrical panels with all edges simply supported is easy to be found [6]. Inclusion of the torsional resistance due to frames attached to the circular edges makes the problem complicated, for which an exact solution is difficult to be identified. It appears, to our knowledge, that no attempt has been made so far to solve it exactly. The exact solutions reported by Wilde et al [7], for instance, are specifically for axially compressed cylindrical panels with three edges simply supported and one edge free. They do not explain the reason why the critical buckling mode is given as a summation of trigonometric and hyperbolic functions, no matter how small or large is the load at which a panel buckles.

Our proposed exact solution, which is the objective of this bachelor thesis, which is based on a Lévy-type procedure [8], has been motivated by the author's need of benchmarks to test the accuracy of a fast tool under development to predict the skin buckling of reinforced cylindrical shells. The solution is stated in a suitable detail, identifying all the function spaces where the critical buckling mode should be sought, and used to highlight the effects of the panel aspect ratio, shallowness and frame torsional resistance on the critical load.

This work is structured in five chapters. Chapter 2 presents some theoretical background to the understanding of the mathematical formulation and solving of the problem. Chapter 3 shows the Donnell's equations and demonstrates that their solutions to the analyzed cylindrical panel may be divided into five different cases. Finally, there are two appendices where it is shown how to derive the torsion moment of the frames and all the solution coefficients. Finally, there is a conclusion followed by the references on which this work is based.

2 Theoretical Foundation

As the problem solving is developed, some mathematical resources, like the ordinary and partial differential equations solving methods and the principle of virtual displacements is used. In order to proceed in the objective of finding a solution to the Donnell's equations, some theoretical background is worth explaining.

Before continuing, let us define the notation to the differentiation operator, for example, related to x as $(\ )_{,x}$, and the one related to y as $(\ )_{,y}$. If we want do get a higher order derivative, we can repete the variable symbol like $(\ )_{,xy}$ or $(\ )_{,xx}$. Generalizing, with derivatives with orders m and n in x and y, respectively, an appropriate notation would be $(\ )_{,x^m y^n}$. That notation does not exclude other well-known notations, like the Leibnitz notation dy/dx or the Lagrange's $f'(x)$, or simply f'.

2.1 Change of Variables

In terms of simplifying the partial differential equations' system, we take hand of a technique consisting in changing the independent variables to another, usually normalized, that can reduce the problem with coefficients. For example, assume the task of changing the initial variables x and y of any function $u(x, y)$ to another two variables ξ e η. To do this, we have to pay attention to the chain rule. For the first differentiation orders, the following equations are valid

$$
\begin{gathered}
u_{,x} = u_{,\xi}\xi_{,x} + u_{,\eta}\eta_{,x} \\
u_{,y} = u_{,\xi}\xi_{,y} + u_{,\eta}\eta_{,y} \\
u_{,xx} = u_{,\xi\xi}\xi_{,x}^2 + 2u_{,\xi\eta}\xi_{,x}\eta_{,x} + u_{,\eta\eta}\eta_{,x}^2 + u_{,\xi}\xi_{,xx} + u_{,\eta}\eta_{,xx} \\
u_{,yy} = u_{,\xi\xi}\xi_{,y}^2 + 2u_{,\xi\eta}\xi_{,y}\eta_{,y} + u_{,\eta\eta}\eta_{,y}^2 + u_{,\xi}\xi_{,yy} + u_{,\eta}\eta_{,yy} \\
u_{,xy} = u_{,\xi\xi}\xi_{,x}\xi_{,y} + u_{,\xi\eta}\left(\xi_{,x}\eta_{,y} + \xi_{,y}\eta_{,x}\right) + u_{,\eta\eta}\eta_{,x}\eta_{,y} + u_{,\xi}\xi_{,xy} + u_{,\eta}\eta_{,xy}
\end{gathered} \tag{2.1}
$$

In this work, we have a particularly simple case, in which we can write

$$\xi = c\,x \qquad \eta = c\,y \tag{2.2}$$

where c is a constant. For this case, we note that the partial derivatives $\xi_{,y}$ and $\eta_{,x}$ are both null, and thus we can write

$$u_{,x} = c\,u_{,\xi} \qquad u_{,y} = c\,u_{,\eta} \tag{2.3}$$

and then generate a more general equation

$$u_{,x^m y^n} = c^{(m+n)} u_{,\xi^m \eta^n}. \tag{2.4}$$

2.2 Solution to Partial Differential Equations

A partial differential equation, also know as PDE, is an equation involving functions with two or more independent variables and depending also on their partial derivatives. These type of functions are widely common in physics and engeneering problems.

In this particular study, the PDE presented on Chapter 3 will be solved supposing an appropriate variable separation. We want this solution to fulfill both boundary values and Donnell's equations.

2.3 Solution to Ordinary Differential Equations

For the sake of pursuing our task to find solutions of a PDE system, we have to take some particular methods of solving ordinary differential equations, also called ODEs. The boundary value problems of this kind of equation have a solution based on the Existency and Uniqueness Theorem, in which we will not enter in details. Before continuing, it is important to know some concepts about diferrential equations.

A definition of ODE is any equation in the form

$$f(x, y, y', y'', \dots) = 0 \tag{2.5}$$

in which we call order of the equation the order of the highest derivative of y. A solution of an ODE can be any function $y = g(x)$ that satisfies the previous equation. A general solution of an ODE is a general linear combination of the solutions of the homogeneous equation summed to the solution of the non-homogeneous equation.

There is a special type of ODE named Linear homogeneous differential equations with constant coefficients. These type of equations can be shown in the form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \tag{2.6}$$

The way to solve this kind of equations consists in seeking for solution of the type $y(x) = e^{sx}$. Substituting this form into the equation (2.6), we then have

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \tag{2.7}$$

We now have a n-order polynomial with n roots that can either be real, imaginary and even be multiple roots. Let us consider a root $\bar{s}$ that could be any of these cases. We then have some rules to define the solution, as follows.

- If $\bar{s} = 0$, than $y(x) = c$
- If $\bar{s}$ is a non-null real simple root,

$$y(x) = e^{\bar{s}x} \tag{2.8}$$

- If $\bar{s}$ is a real multiple root of order k

$$y_1 = e^{\bar{s}x}, y_2 = xe^{\bar{s}x}, y_3 = x^2 e^{\bar{s}x}, \dots , y_k = x^{k-1}e^{\bar{s}x} \tag{2.9}$$

- If $\bar{s}$ is a purely imaginary root of the type $\bar{s} = \pm i\beta$

$$y_1 = \sin \beta x \qquad y_2 = \cos \beta x$$

If $\bar{s}$ is a complex root of the type $\bar{s} = \alpha \pm i\beta$

$$y_1 = e^{\alpha x}\sin \beta x \qquad y_2 = e^{\alpha x}\cos \beta x$$

If $\bar{s}$ is a complex root of the type $\bar{s} = \pm\alpha \pm i\beta$ you can also write

$$y_1 = \cosh \alpha x \sin \beta x \qquad y_2 = cosh\ \alpha x \cos \beta x,$$

$$y_3 = \sinh \alpha x \sin \beta x \qquad y_4 = sinh\ \alpha x \cos \beta x \tag{2.10}$$

- If $\bar{s}$ is a multiple complex root, the rule is similar to the real multiple root case.

Note that the solution of a boundary condition problem can be a linear combination of all of these particular solutions, with constants being determined by the boundary conditions applied to them.

2.4 Principle of Virtual Displacements

For the sake of determining how the torsion moment on the frames attached to the circular edges of the panel behave, we take another mathematician trick widely used in structural engineering: the calculus of variation and the principle of virtual displacements.

First, we have to define a physical quantity representing an energy an object has to receive to move from a place to another one: the work. In terms of the displacement vector, an infinitesimal work done by a force which moves this body is

$$dW = \int \vec{F}.d\vec{u} \tag{2.18}$$

Note that the work done by this force depends on the scalar product of the displacement done by the object at this infinitesimal variation of time and the vector force itself. Thus, the infinitesimal quantity dW is a scalar number, which can assume any real value.

We can observe that a particle, to have its static equilibrium disturbed, needs a positive value of work done by an external force to be transformed in kinetic energy. At this point, we have to introduce a new mathematical entity called virtual displacement, denoted by $\delta\vec{u}$, representing any materially possible arbitrary variation of space position done by the object. The term virtual used, rather than real, takes place because no displacement has actually been done, since there was no passage of time. This concept is very important to mechanics of structures. The principle of virtual displacements says:

"A particle is in equilibrium if, and only if, the sum of the virtual works done by external forces applied to it is null for virtual displacements*"*.

Another statement of this principle to a deformable solid, which is mainly the type of body studied by the structural engineers, is

"A deformable solid is in equilibrium if, and only if, the sum of the virtual work done by external and internal forces is *null for virtual displacements"*.

3 Formulation of the Problem

Figure 3.1(a) depicts a circular cylindrical panel of radius R, length a, width b and thickness h, subjected to a uniform distributed axial compressive force p per unit length and referred to a set of orthogonal curvilinear coordinates xyz placed in the panel midsurface. The panel buckling is supposed to be described by the linearized Donnell's equations

$$\nabla^4 u = -\frac{\nu}{R} w_{,xxx} + \frac{1}{R} w_{,xyy}$$

$$\nabla^4 v = -\frac{2+\nu}{R} w_{,xxy} - \frac{1}{R} w_{,yyy}$$

$$D\nabla^8 w + \frac{Eh}{R^2} w_{,xxxx} + p\nabla^4 w_{,xx} = 0 \tag{3.1}$$

with prebuckling rotations neglected [3].The quantities u, v and w are the midsurface displacements in the x, y and z directions, E is the Young's modulus, ν is the Poisson's ratio, $D = Eh^3/12(1-\nu^2)$ defines the panel bending rigidity, ∇^8 denotes two successive applications of the two-dimensional biharmonic operator $\nabla^4(\ \ ) = (\ \ )_{,xxxx} + 2(\ \ )_{,xxyy} + (\ \ )_{,yyyy}$ and a comma followed by x (or y) indicates differentiation with respect to x (or y). The boundary conditions (see Fig. 3.1(b)) to be applied differ from the classical simply supported ones in the sense that the torsion resisted by the frames attached to the panel circular edges are also taken into account:

$$N_x = 0 \qquad v = 0 \qquad w = 0 \qquad M_x = \bar{M} \qquad \text{at } x = \pm\frac{a}{2}$$

$$u = 0 \qquad N_y = 0 \qquad w = 0 \qquad M_y = 0 \qquad \text{at } y = 0, b. \tag{3.2}$$

The in-plane forces N_x, N_y and the bending moments M_x, M_y are related to the midsurface displacements by means of

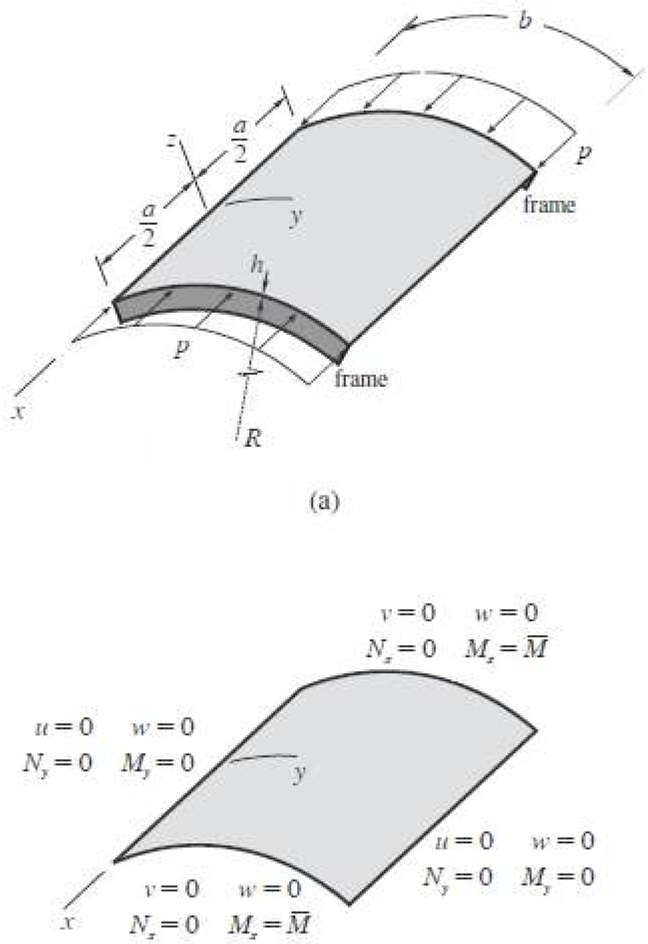

Fig. 3.1. Cylindrical panel: (a) geometry and loading; (b) boundary conditions.

$$N_x = \frac{Eh}{1-\nu^2}\left[u_{,x} + \nu\left(v_{,y} + \frac{w}{R}\right)\right] \qquad N_y = \frac{Eh}{1-\nu^2}\left(v_{,y} + \frac{w}{R} + \nu u_{,x}\right)$$

$$M_x = -D\left(w_{,xx} + \nu w_{,yy}\right) \qquad M_y = -D\left(w_{,yy} + \nu w_{,xx}\right) \tag{3.3}$$

and the external moments

$$\begin{aligned}\bar{M}\left(\pm\frac{a}{2}, y\right) = &\mp\frac{E_f I_f}{R}\left(u_{,yy}\left(\pm\frac{a}{2}, y\right) - \frac{1}{R}w_{,x}\left(\pm\frac{a}{2}, y\right)\right)\\ &\mp G_f J_f\left(\frac{1}{R}u_{,yy}\left(\pm\frac{a}{2}, y\right) + w_{,xyy}\left(\pm\frac{a}{2}, y\right)\right)\\ &\pm E_f \Gamma_f\left(\frac{1}{R}u_{,yyyy}\left(\pm\frac{a}{2}, y\right) + w_{,xyyyy}\left(\pm\frac{a}{2}, y\right)\right)\end{aligned} \tag{3.4}$$

exerted by the frames on the panel edges $x = \pm a/2$ prevent them from rotating freely (see Appendix A). The frames have torsional and warping constants J_f and Γ_f, area moment of inertia of the cross section about the z axis given by I_f, material with Young's modulus E_f and shear modulus G_f, and are twisted by an amount $-w_{,x}$. The first term in $\bar{M}$ is associated with the frame out-of-plane bending, while the second and third terms are associated with Saint-Venant and warping torsions, respectively.

An exact solution of (3.1) subjected to (3.22) can be accomplished most conveniently by introducing the nondimentional coordinates

$$\xi = \sqrt{\frac{2Eh}{p_{cl}}}\frac{x}{R} \qquad \eta = \sqrt{\frac{2Eh}{p_{cl}}}\frac{y}{R} \tag{3.5}$$

suggested by Nachbar [9],where

$$p_{cl} = \frac{Eh^2}{R\sqrt{3(1-\nu^2)}}. \tag{3.6}$$

The classical value p_{cl} identifies the minimum buckling load p that a simply supported panel could ever achieved [6], according to the adopted linear buckling analysis. Referring to these new coordinates, the system of equations (3.1) becomes

$$\nabla^4 u = -\nu w_{,\xi\xi\xi} + w_{,\xi\eta\eta}$$

$$\nabla^4 v = -(2+\nu) w_{,\xi\xi\eta} - w_{,\eta\eta\eta}$$

$$\nabla^8 w + w_{,\xi\xi\xi\xi} + 2\rho\nabla^4 w_{,\xi\xi} = 0 \tag{3.7}$$

while the set of boundary conditions (3.2) reads

$$\begin{array}{lllll} N_\xi = 0 & v = 0 & w = 0 & M_\xi = \bar{M} & \text{at } \xi = \pm\,\xi_0 \\ u = 0 & N_\eta = 0 & w = 0 & M_\eta = 0 & \text{at } \eta = 0, \eta_0 \end{array} \tag{3.8}$$

with

$$\rho = \frac{p}{p_{cl}} \qquad \xi_0 = \sqrt{\frac{2Eh}{p_{cl}}}\frac{a}{2R} \qquad \eta_0 = \sqrt{\frac{2Eh}{p_{cl}}}\frac{b}{R}. \tag{3.9}$$

In (3.7) and (3.8), the operators ∇^4 and ∇^8 are written in the new coordinates ξ and η. The quantities

$$u(\xi,\eta) = \sqrt{\frac{2Eh}{p_{cl}}}\frac{u(x,y)}{R} \qquad v(\xi,\eta) = \sqrt{\frac{2Eh}{p_{cl}}}\frac{v(x,y)}{R} \qquad w(\xi,\eta) = \frac{w(x,y)}{R} \quad (3.10)$$

now represent nondimensional displacement parameters and

$$N_\xi = \frac{Eh}{1-\nu^2}\left(u_{,\xi} + \nu v_{,\eta} + \nu w\right) \qquad N_\eta = \frac{Eh}{1-\nu^2}\left(\nu u_{,\xi} + v_{,\eta} + w\right)$$

$$M_\xi = -\frac{D}{R}\frac{2Eh}{p_{cl}}\left(w_{,\xi\xi} + \nu w_{,\eta\eta}\right) \qquad M_\eta = -\frac{D}{R}\frac{2Eh}{p_{cl}}\left(\nu w_{,\xi\xi} + w_{,\eta\eta}\right)$$

$$\begin{aligned}\bar{M}(\pm\xi_0,\eta) = &\mp\frac{E_f I_f}{R^2}\sqrt{\frac{2Eh}{p_{cl}}}\left(u_{,\eta\eta}(\pm\xi_0,\eta) - w_{,\xi}(\pm\xi_0,\eta)\right) \\ &\mp\frac{G_f J_f}{R^2}\sqrt{\frac{2Eh}{p_{cl}}}\left(u_{,\eta\eta}(\pm\xi_0,\eta) + \frac{2Eh}{p_{cl}}w_{,\xi\eta\eta}(\pm\xi_0,\eta)\right) \\ &\pm\frac{E_f\Gamma_f}{R^4}\left(\frac{2Eh}{p_{cl}}\right)^{3/2}\left(u_{,\eta\eta\eta\eta}(\pm\xi_0,\eta) + \frac{2Eh}{p_{cl}}w_{,\xi\eta\eta\eta\eta}(\pm\xi_0,\eta)\right). \qquad (3.11)\end{aligned}$$

4 Results

Given the problem formulation, we now are able to find exact solutions by solving its associated equations. A general buckling mode satisfying the simply supported boundary conditions on edges $\eta = 0, \eta_0$ can be assumed in the form

$$u(\xi,\eta) = U(\xi)\sin k\eta \qquad v(\xi,\eta) = V(\xi)\cos k\eta \qquad w(\xi,\eta) = W(\xi)\sin k\eta \quad (4.1)$$

with

$$k = \frac{n\pi R}{b}\sqrt{\frac{p_{cl}}{2Eh}} \tag{4.2}$$

and the integer n standing for the number of half-waves in the circumferential direction. The functions $U(\xi)$, $V(\xi)$ and $W(\xi)$ must be obtained so that the mode fulfills the conditions required by the supports at $\xi = \pm\xi_0$ and satisfies (3.7).

After substitution of (4.1), the boundary conditions (3.8) on edges $\xi = \pm\xi_0$ hold for every $0 < \eta < \eta_0$ if

$$U_{,\xi}(\pm\xi_0) = 0 \qquad V(\pm\xi_0) = 0 \qquad W(\pm\xi_0) = 0$$

$$\frac{D}{R}W_{,\xi\xi}(\pm\xi_0) \pm \sqrt{\frac{p_{cl}}{2Eh}}\left(\frac{E_f I_f}{R^2} + \frac{n^2\pi^2}{b^2}G_f J_f + \frac{n^4\pi^4}{b^4}E_f\Gamma_f\right)W_{,\xi}(\pm\xi_0)$$
$$\pm\frac{n^2\pi^2}{b^2}\left(\frac{p_{cl}}{2Eh}\right)^{\frac{3}{2}}\left(E_f I_f + G_f J_f + \frac{n^2\pi^2}{b^2}E_f\Gamma_f\right)U(\pm\xi_0) = 0. \tag{4.3}$$

On the other hand, the equation obtained after substitution of $w(\xi,\eta)$ into the third of equations (3.7) holds for every point (ξ,η) of the domain for nontrivial w (i.e., $W \neq 0$) if

$$\frac{d^8W}{d\xi^8} - 2(2k^2-\rho)\frac{d^6W}{d\xi^6} + (6k^4 - 4\rho k^2 + 1)\frac{d^4W}{d\xi^4}$$
$$-2k^4(2k^2-\rho)\frac{d^2W}{d\xi^2} + k^8W = 0. \tag{4.4}$$

Particular solutions of this homogeneous linear differential equation are in the form $e^{s\xi}$, where s denotes a root of the algebraic equation

$$s^8 - 2(2k^2 - \rho)s^6 + (6k^4 - 4\rho k^2 + 1)s^4 - 2k^4(2k^2 - \rho)s^2 + k^8 = 0. \quad (4.5)$$

The roots of (4.5) can be easily found by rewriting the equation as

$$\left(\frac{s^2 - k^2}{s}\right)^4 + 2\rho\left(\frac{s^2 - k^2}{s}\right)^2 + 1 = 0 \quad (4.6)$$

from which

$$s = \frac{\pm i\sqrt{\lambda_i} \pm \sqrt{4k^2 - \lambda_i}}{2} \quad (4.7)$$

where

$$\lambda_1 = \rho - \sqrt{\rho^2 - 1} \qquad \lambda_2 = \rho + \sqrt{\rho^2 - 1} \quad (4.8)$$

and $i = \sqrt{-1}$ denotes the imaginary unit. It is clear from physical considerations that the buckling will always take place for $\rho > 1$ due to the attachment of frames to the panel edges $\xi = \pm\xi_0$. In view of this evidence, the parameters $\lambda_2 > \lambda_1 > 0$.

The presence of the constant term k^8 ($k > 0$) in (4.5) and the property $\lambda_i > 0$ anticipate that zero and real roots do not exist. Depending on the values of λ_i and k, the roots may be grouped according to the five cases in the sequel.

4.1 Case I: $0 < k^2 < \lambda_1/4$

All the roots are distinct and purely imaginary:

$$s_1 = -s_5 = i\gamma_1 \qquad s_2 = -s_6 = i\gamma_2 \qquad s_3 = -s_7 = i\gamma_3 \qquad s_4 = -s_8 = i\gamma_4 \quad (4.9)$$

with

$$\left.\begin{matrix}\gamma_1\\ \gamma_2\end{matrix}\right\} = \frac{\sqrt{\lambda_1} \pm \sqrt{\lambda_1 - 4k^2}}{2} \qquad \left.\begin{matrix}\gamma_3\\ \gamma_4\end{matrix}\right\} = \frac{\sqrt{\lambda_2} \pm \sqrt{\lambda_2 - 4k^2}}{2}. \quad (4.10)$$

The solution of (4.4) may then be taken in the form

$$\begin{aligned} W(\xi) &= W_1 \sin\gamma_1\xi + W_2 \cos\gamma_1\xi + W_3 \sin\gamma_2\xi + W_4 \cos\gamma_2\xi \\ &\quad + W_5 \sin\gamma_3\xi + W_6 \cos\gamma_3\xi + W_7 \sin\gamma_4\xi + W_8 \cos\gamma_4\xi \end{aligned} \quad (4.11)$$

where W_i are arbitrary constants.

Because the yz plane is a symmetry plane for the structure (see Fig. 3.1(a)), the buckling modes can be separated into two distinct symmetry classes, which may be readily identified by the shape of $W(\xi)$. The modes may be classified by whether the displacement component w is symmetric or antisymmetric with respect to the yz plane. The displacement components u and v will then also have appropriate symmetries. This separation not only aids in identifying and classifying the buckling mode, but also reduces the eigenvalue problem to two distinct problems with smaller determinants to be evaluated. Using the subscripts "s" and "a" to refer to symmetric and antisymmetric parts, Eq. (4.11) is split into

$$W_s(\xi) = W_{1s}\cos\gamma_1\xi + W_{2s}\cos\gamma_2\xi + W_{3s}\cos\gamma_3\xi + W_{4s}\cos\gamma_4\xi$$

$$W_a(\xi) = W_{1a}\sin\gamma_1\xi + W_{2a}\sin\gamma_2\xi + W_{3a}\sin\gamma_3\xi + W_{4a}\sin\gamma_4\xi \qquad (4.12)$$

where W_{is} and W_{ia} are appropriate redefinitions of W_i. From (3.7), (4.1) and (4.12), the functions $U(\xi)$ and $V(\xi)$ may also be split into

$$U_s(\xi) = W_{1s}u_{1s}\sin\gamma_1\xi + W_{2s}u_{2s}\sin\gamma_2\xi + W_{3s}u_{3s}\sin\gamma_3\xi + W_{4s}u_{4s}\sin\gamma_4\xi$$

$$U_a(\xi) = W_{1a}u_{1a}\cos\gamma_1\xi + W_{2a}u_{2a}\cos\gamma_2\xi + W_{3a}u_{3a}\cos\gamma_3\xi + W_{4a}u_{4a}\cos\gamma_4\xi$$

$$V_s(\xi) = W_{1s}v_{1s}\cos\gamma_1\xi + W_{2s}v_{2s}\cos\gamma_2\xi + W_{3s}v_{3s}\cos\gamma_3\xi + W_{4s}v_{4s}\cos\gamma_4\xi$$

$$V_a(\xi) = W_{1a}v_{1a}\sin\gamma_1\xi + W_{2a}v_{2a}\sin\gamma_2\xi + W_{3a}v_{3a}\sin\gamma_3\xi + W_{4a}v_{4a}\sin\gamma_4\xi \quad (4.13)$$

The coefficients u_{is}, u_{ia}, v_{is} and v_{ia} are detailed in Appendix B.

Introduction of the symmetric (or antisymmetric) displacement components (4.12) and (4.13) into (4.3) for $\xi = \xi_0$ yields the homogeneous system of equations

$$\mathbf{KW} = \mathbf{0}, \qquad (4.14)$$

where the vector $\mathbf{W}$ collects W_{is} (or W_{ia}) and the matrix $\mathbf{K}$ is given in Appendix B. Each root ρ of the equation $\det\mathbf{K} = 0$ represents a buckling load.

Since similar expressions to (4.14) hold for the remaining cases, for which the matrix $\mathbf{K}$ is also listed in Appendix B, only the functions $U(\xi)$, $V(\xi)$ and $W(\xi)$ associated with the solution will be summarized next.

4.2 Case II: $k^2 = \lambda_1/4$

As the parameters $\gamma_1 = \gamma_2$ in (4.10), the roots (4.9) reduces to

$$s_1 = s_2 = -s_5 = -s_6 = i\gamma_1 \qquad s_3 = -s_7 = i\gamma_3 \qquad s_4 = -s_8 = i\gamma_4. \tag{4.15}$$

The solutions (4.12) and (4.13) must be modified to account for the repeated roots $s_1 = s_2$ and $s_5 = s_6$:

$$W_s(\xi) = W_{1s}\cos\gamma_1\xi + W_{2s}\xi\sin\gamma_1\xi + W_{3s}\cos\gamma_3\xi + W_{4s}\cos\gamma_4\xi$$

$$W_a(\xi) = W_{1a}\sin\gamma_1\xi + W_{2a}\xi\cos\gamma_1\xi + W_{3a}\sin\gamma_3\xi + W_{4a}\sin\gamma_4\xi$$

$$U_s(\xi) = (W_{1s}u_{1s} + W_{2s}u_{2s})\sin\gamma_1\xi + W_{2s}\bar{u}_{2s}\xi\cos\gamma_1\xi + W_{3s}u_{3s}\sin\gamma_3\xi + W_{4s}u_{4s}\sin\gamma_4\xi$$

$$U_a(\xi) = (W_{1a}u_{1a} + W_{2a}u_{2a})\cos\gamma_1\xi + W_{2a}\bar{u}_{2a}\xi\sin\gamma_1\xi + W_{3a}u_{3a}\cos\gamma_3\xi + W_{4a}u_{4a}\cos\gamma_4\xi$$

$$V_s(\xi) = (W_{1s}v_{1s} + W_{2s}v_{2s})\cos\gamma_1\xi + W_{2s}\bar{v}_{2s}\xi\sin\gamma_1\xi + W_{3s}v_{3s}\cos\gamma_3\xi + W_{4s}v_{4s}\cos\gamma_4\xi$$

$$V_a(\xi) = (W_{1a}v_{1a} + W_{2a}v_{2a})\sin\gamma_1\xi + W_{2a}\bar{v}_{2a}\xi\cos\gamma_1\xi + W_{3a}v_{3a}\sin\gamma_3\xi + W_{4a}v_{4a}\sin\gamma_4\xi \tag{4.16}$$

The coefficients u_{is}, $\bar{u}_{is}$, u_{ia}, $\bar{u}_{ia}$, v_{is}, $\bar{v}_{is}$, v_{ia}, $\bar{v}_{ia}$ are detailed in Appendix B.

4.3 Case III: $\lambda_1/4 < k^2 < \lambda_2/4$

All the roots are distinct. Roots s_1, s_2, s_5, s_6 are the complex roots

$$\left.\begin{matrix} s_1 = -s_5 \\ s_2 = -s_6 \end{matrix}\right\} = \alpha_1 \pm i\beta_1, \tag{4.17}$$

with

$$\alpha_1 = \frac{\sqrt{4k^2 - \lambda_1}}{2} \qquad \beta_1 = \frac{\sqrt{\lambda_1}}{2}, \tag{4.18}$$

and roots s_3, s_4, s_7, s_8 are the purely imaginary roots identified in Case I. Splitting the solution of (4.4) into symmetric and antisymmetric parts as before,

$$W_s(\xi) = W_{1s} \sinh \alpha_1\xi \sin \beta_1\xi + W_{2s} \cosh \alpha_1\xi \cos \beta_1\xi + W_{3s} \cos \gamma_3\xi + W_{4s} \cos \gamma_4\xi$$

$$W_a(\xi) = W_{1a} \sinh \alpha_1\xi \cos \beta_1\xi + W_{2a} \cosh \alpha_1\xi \sin \beta_1\xi + W_{3a} \sin \gamma_3\xi + W_{4a} \sin \gamma_4\xi$$

$$U_s(\xi) = (W_{1s}u_{1s}+W_{2s}u_{2s}) \sinh \alpha_1\xi \cos \beta_1\xi + (W_{1s}\bar{u}_{1s}+W_{2s}\bar{u}_{2s}) \cosh \alpha_1\xi \sin \beta_1\xi + W_{3s}u_{3s} \sin \gamma_3\xi + W_{4s}u_{4s} \sin \gamma_4\xi$$

$$U_a(\xi) = (W_{1a}u_{1a}+W_{2a}u_{2a}) \sinh \alpha_1\xi \sin \beta_1\xi + (W_{1a}\bar{u}_{1a}+W_{2a}\bar{u}_{2a}) \cosh \alpha_1\xi \cos \beta_1\xi + W_{3a}u_{3a} \cos \gamma_3\xi + W_{4a}u_{4a} \cos \gamma_4\xi$$

$$V_s(\xi) = (W_{1s}v_{1s}+W_{2s}v_{2s}) \sinh \alpha_1\xi \sin \beta_1\xi + (W_{1s}\bar{v}_{1s}+W_{2s}\bar{v}_{2s}) \cosh \alpha_1\xi \cos \beta_1\xi + W_{3s}v_{3s} \cos \gamma_3\xi + W_{4s}v_{4s} \cos \gamma_4\xi$$

$$V_a(\xi) = (W_{1a}v_{1a}+W_{2a}v_{2a}) \sinh \alpha_1\xi \cos \beta_1\xi + (W_{1a}\bar{v}_{1a}+W_{2a}\bar{v}_{2a}) \cosh \alpha_1\xi \sin \beta_1\xi + W_{3a}v_{3a} \sin \gamma_3\xi + W_{4a}v_{4a} \sin \gamma_4\xi. \tag{4.19}$$

The coefficients u_{is}, $\bar{u}_{is}$, u_{ia}, $\bar{u}_{ia}$, v_{is}, $\bar{v}_{is}$, v_{ia}, $\bar{v}_{ia}$ are detailed in Appendix B.

4.4 Case IV: $k^2 = \lambda_2/4$

Roots s_1, s_2, s_5, s_6 are the complex roots identified in Case III, whereas s_3, s_4, s_7, s_8 reduce to the purely imaginary roots

$$s_3 = s_4 = -s_7 = -s_8 = i\gamma_3. \tag{4.20}$$

In order to account for the repeated roots $s_3 = s_4$ and $s_7 = s_8$, the solution (4.19) must be modified to read

$$W_s(\xi) = W_{1s} \sinh \alpha_1\xi \sin \beta_1\xi + W_{2s} \cosh \alpha_1\xi \cos \beta_1\xi + W_{3s} \cos \gamma_3\xi + W_{4s}\xi \sin \gamma_3\xi$$

$$W_a(\xi) = W_{1a}\sinh\alpha_1\xi\cos\beta_1\xi + W_{2a}\cosh\alpha_1\xi\sin\beta_1\xi + W_{3a}\sin\gamma_3\xi + W_{4a}\xi\cos\gamma_3\xi$$

$$\begin{aligned} U_s(\xi) = {} & (W_{1s}u_{1s}+W_{2s}u_{2s})\sinh\alpha_1\xi\cos\beta_1\xi + (W_{1s}\bar{u}_{1s}+W_{2s}\bar{u}_{2s})\cosh\alpha_1\xi\sin\beta_1\xi \\ & + (W_{3s}u_{3s} + W_{4s}u_{4s})\sin\gamma_3\xi + W_{4s}\bar{u}_{4s}\xi\cos\gamma_3\xi \end{aligned}$$

$$\begin{aligned} U_a(\xi) = {} & (W_{1a}u_{1a}+W_{2a}u_{2a})\sinh\alpha_1\xi\sin\beta_1\xi + (W_{1a}\bar{u}_{1a}+W_{2a}\bar{u}_{2a})\cosh\alpha_1\xi\cos\beta_1\xi \\ & + (W_{3a}u_{3a} + W_{4a}u_{4a})\cos\gamma_3\xi + W_{4a}\bar{u}_{4a}\xi\sin\gamma_3\xi \end{aligned}$$

$$\begin{aligned} V_s(\xi) = {} & (W_{1s}v_{1s}+W_{2s}v_{2s})\sinh\alpha_1\xi\sin\beta_1\xi + (W_{1s}\bar{v}_{1s}+W_{2s}\bar{v}_{2s})\cosh\alpha_1\xi\cos\beta_1\xi \\ & + (W_{3s}v_{3s} + W_{4s}v_{4s})\cos\gamma_3\xi + W_{4s}\bar{v}_{4s}\xi\sin\gamma_3\xi \end{aligned}$$

$$\begin{aligned} V_a(\xi) = {} & (W_{1a}v_{1a}+W_{2a}v_{2a})\sinh\alpha_1\xi\cos\beta_1\xi + (W_{1a}\bar{v}_{1a}+W_{2a}\bar{v}_{2a})\cosh\alpha_1\xi\sin\beta_1\xi \\ & + (W_{3a}v_{3a} + W_{4a}v_{4a})\sin\gamma_3\xi + W_{4a}\bar{v}_{4a}\xi\cos\gamma_3\xi. \end{aligned} \tag{4.21}$$

The coefficients u_{is}, $\bar{u}_{is}$, u_{ia}, $\bar{u}_{ia}$, v_{is}, $\bar{v}_{is}$, v_{ia}, $\bar{v}_{ia}$ are detailed in Appendix B.

4.5 Case V: $\lambda_2/4 < k^2$

All the roots are distinct and complex:

$$\left.\begin{matrix} s_1 = -s_5 \\ s_2 = -s_6 \end{matrix}\right\} = \alpha_1 \pm i\beta_1 \qquad \left.\begin{matrix} s_3 = -s_7 \\ s_4 = -s_8 \end{matrix}\right\} = \alpha_2 \pm i\beta_2 \tag{4.22}$$

with

$$\alpha_i = \frac{\sqrt{4k^2 - \lambda_i}}{2} \qquad \beta_i = \frac{\sqrt{\lambda_i}}{2}. \tag{4.23}$$

The symmetric and antisymmetric solutions of (4.4) are

$$\begin{aligned} W_s(\xi) = {} & W_{1s}\sinh\alpha_1\xi\sin\beta_1\xi + W_{2s}\cosh\alpha_1\xi\cos\beta_1\xi + W_{3s}\sinh\alpha_2\xi\sin\beta_2\xi \\ & + W_{4s}\cosh\alpha_2\xi\cos\beta_2\xi \end{aligned}$$

$$\begin{aligned} W_a(\xi) = {} & W_{1a}\sinh\alpha_1\xi\cos\beta_1\xi + W_{2a}\cosh\alpha_1\xi\sin\beta_1\xi + W_{3a}\sinh\alpha_2\xi\cos\beta_2\xi \\ & + W_{4a}\cosh\alpha_2\xi\sin\beta_2\xi \end{aligned}$$

$$U_s(\xi) = (W_{1s}u_{1s}+W_{2s}u_{2s})\sinh\alpha_1\xi\cos\beta_1\xi + (W_{1s}\bar{u}_{1s}+W_{2s}\bar{u}_{2s})\cosh\alpha_1\xi\sin\beta_1\xi$$
$$+ (W_{3s}u_{3s}+W_{4s}u_{4s})\sinh\alpha_2\xi\cos\beta_2\xi + (W_{3s}\bar{u}_{3s}+W_{4s}\bar{u}_{4s})\cosh\alpha_2\xi\sin\beta_2\xi$$

$$U_a(\xi) = (W_{1a}u_{1a}+W_{2a}u_{2a})\sinh\alpha_1\xi\sin\beta_1\xi + (W_{1a}\bar{u}_{1a}+W_{2a}\bar{u}_{2a})\cosh\alpha_1\xi\cos\beta_1\xi$$
$$+ (W_{3a}u_{3a}+W_{4a}u_{4a})\sinh\alpha_2\xi\sin\beta_2\xi + (W_{3a}\bar{u}_{3a}+W_{4a}\bar{u}_{4a})\cosh\alpha_2\xi\cos\beta_2\xi$$

$$V_s(\xi) = (W_{1s}v_{1s}+W_{2s}v_{2s})\sinh\alpha_1\xi\sin\beta_1\xi + (W_{1s}\bar{v}_{1s}+W_{2s}\bar{v}_{2s})\cosh\alpha_1\xi\cos\beta_1\xi$$
$$+ (W_{3s}v_{3s}+W_{4s}v_{4s})\sinh\alpha_2\xi\sin\beta_2\xi + (W_{3s}\bar{v}_{3s}+W_{4s}\bar{v}_{4s})\cosh\alpha_2\xi\cos\beta_2\xi$$

$$V_a(\xi) = (W_{1a}v_{1a}+W_{2a}v_{2a})\sinh\alpha_1\xi\cos\beta_1\xi + (W_{1a}\bar{v}_{1a}+W_{2a}\bar{v}_{2a})\cosh\alpha_1\xi\sin\beta_1\xi$$
$$+ (W_{3a}v_{3a}+W_{4a}v_{4a})\sinh\alpha_2\xi\cos\beta_2\xi + (W_{3a}\bar{v}_{3a}+W_{4a}\bar{v}_{4a})\cosh\alpha_2\xi\sin\beta_2\xi. \quad (4.24)$$

The coefficients u_{is}, $\bar{u}_{is}$, u_{ia}, $\bar{u}_{ia}$, v_{is}, $\bar{v}_{is}$, v_{ia}, $\bar{v}_{ia}$ are detailed in Appendix B.

5 Conclusions

It has been demonstrated, by means of Lévy's method, that exact solutions can be found for the buckling of axially-compressed cylindrical panels described by the linearized Donnell's equations with simply supported conditions modified by frames attached to the circular edges. Five different function spaces, in which the critical buckling mode should be sought, are identified. Each of such spaces is defined by the relative values of the parameters k^2, λ_1 and λ_2.

The process of finding an exact solution leads to the eigenvalue problem $\boldsymbol{K}\,\boldsymbol{W} = 0$, whose matrix $\boldsymbol{K}$ is a function of the load parameter ρ. The present work finished at this point, after explicitly stating the matrix $\boldsymbol{K}$ for each of the five possible function spaces. As a further step, the eigenvalue problem must be solved to identify the critical load and corresponding buckling mode.

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Appendix A – Torsion of the Frame

Equation (3.4) is not trivial, so this appendix will derive it, starting from the equilibrium conditions of a circular bar and progressing through the principleof virtual displacements.

Let us start writing for the panel some equilibrium equations, what is easily done by newtonian's mechanics. Let us write the equilibrium for an infinitesimal element shown in Fig. A.1.

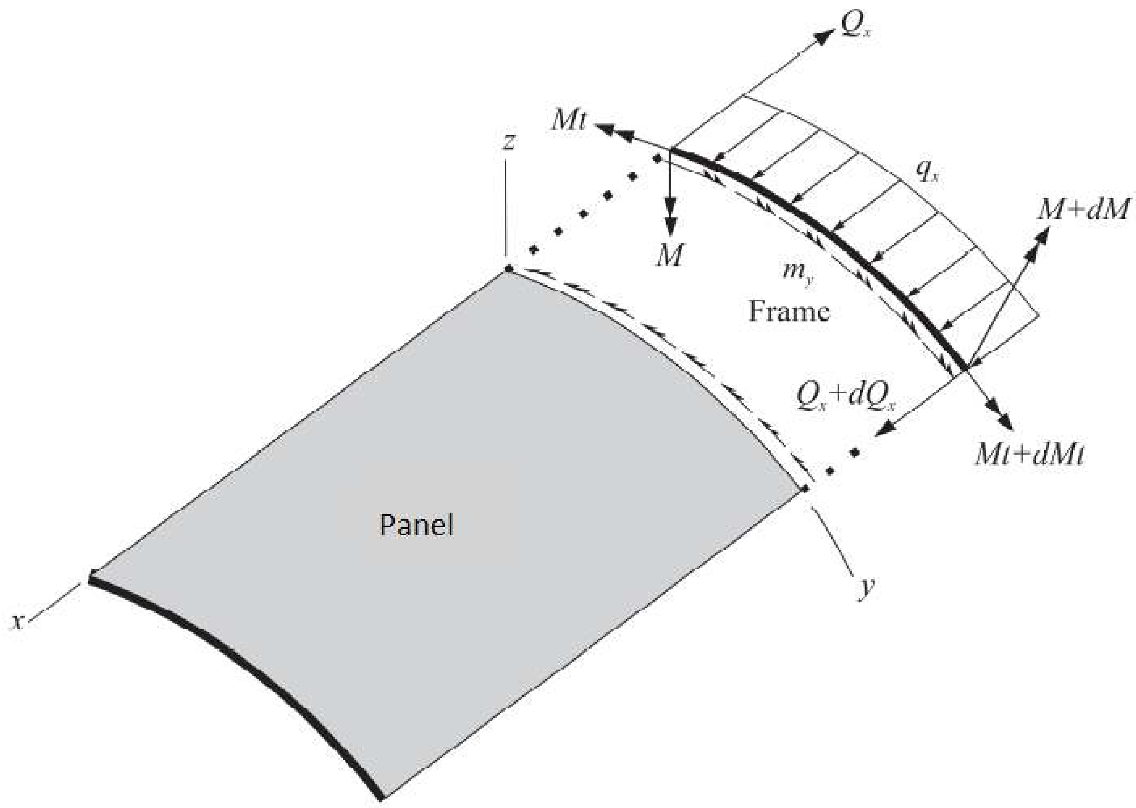

Figure A.1 – Cylindrical Panel with an infinitesimal element of frame.

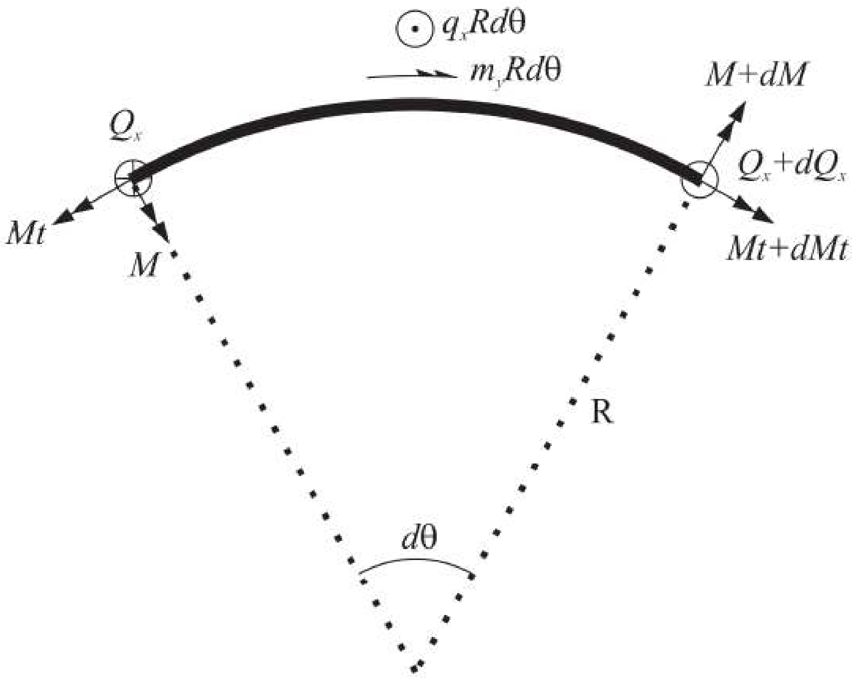

Figure A.2 – Infinitesimal element of frame

From the equilibrium in the direction x

$$-Q_x + Q_x + dQ_x + q_x R d\theta = 0 \qquad R d\theta = dy$$

$$\frac{dQ_x}{dy} + q_x = 0 \tag{A.1}$$

From the moment equilibrium about z axis

$$-M_t \sin\frac{d\theta}{2} - (M_t + dM_t)\sin\frac{d\theta}{2} - M\cos\frac{d\theta}{2} + (M + dM)\cos\frac{d\theta}{2} - Q_x R d\theta = 0$$

$$-2M_t \sin\frac{d\theta}{2} - dM_t \sin\frac{d\theta}{2} + dM\cos\frac{d\theta}{2} - Q_x R d\theta = 0$$

$$-M_t \sin d\theta - dM_t \sin\frac{d\theta}{2} + dM - Q_x R d\theta$$

$$-M_t d\theta + dM_t - Q_x R d\theta = 0$$

$$-\frac{M_t}{R} + \frac{dM_t}{Rd\theta} - Q_x = 0$$

$$-\frac{M_t}{R} + \frac{dM_t}{dy} - Q_x = 0. \tag{A.2}$$

From the moment equilibrium about y axis

$$-M_t \cos\frac{d\theta}{2} + (M_t + dM_t)\cos\frac{d\theta}{2} + M\sin\frac{d\theta}{2} + (M + dM)\sin\frac{d\theta}{2} + m_y R d\theta = 0$$

$$dM_t \cos\frac{d\theta}{2} + 2M\cos\frac{d\theta}{2} + dM\sin\frac{d\theta}{2} + m_y R d\theta = 0$$

$$dM_t \cos\frac{d\theta}{2} Md\theta + dM\frac{d\theta}{2} + m_y R d\theta = 0$$

$$dM_t + Md\theta + m_y R d\theta = 0$$

$$\frac{dM_t}{Rd\theta} + \frac{M}{R} + m_y = 0$$

$$m_y = -\frac{M}{R} - \frac{dM_t}{dy} \tag{A.3}$$

Based on results of straight bars, we write

$$M_t = \left(M_{sv} - \frac{dM_w}{dy}\right) \tag{A.4}$$

in order to reduce the equilibrium equations to

$$Q_{x,y} + q_x = 0$$

$$-\frac{M_{sv}}{R} + \frac{M_{w,y}}{R} + M_{,y} - Q_x = 0$$

$$-\frac{M}{R} - M_{sv,y} + M_{w,yy} - m_y = 0 \tag{A.5}$$

Equation (A.5) may by related to the principle of virtual displacements by means of

$$\int_0^b \left[(Q_{x,y} + q_x)\delta u - \left(-\frac{M_{sv}}{R} + \frac{M_{w,y}}{R} + M_{,y} - Q_x\right)\delta\theta_z \right.$$
$$\left. + \left(\frac{M}{R} + M_{sv,y} - M_{w,yy} + m_y\right)\delta\theta_y\right] dy + \cdots = 0. \tag{A.6}$$

Let us use this expression to discover the deformation-displacement relations which are consistent to the equilibrium equations. Using integration by parts

$$\int_0^b Q_{x,y}\delta u\, dy = Q_x \delta u|_0^b - \int_0^b Q_x \delta u_{,y}\, dy$$

$$\int_0^b M_{w,yy}\delta\theta_y\, dy = M_{w,y}\delta\theta_y|_0^b - \int_0^b M_{w,y}\delta\theta_{y,y}\, dy$$

$$= M_{w,y}\delta\theta_y|_0^b - M_w\delta\theta_{y,y}|_0^b + \int_0^b M_w \delta\theta_{y,yy}\, dy \tag{A.7}$$

we obtain

$$\int_0^b \left[\left(-Q_x\delta u_{,y} + q_x\delta u\right) + \left(-\frac{M_{sv}}{R}\delta\theta_z - \frac{M_w}{R}\delta\theta_{z,y} + M\delta\theta_{z,y} - Q_x\delta\theta_z\right)\right.$$

$$\left. + \left(\frac{M}{R}\delta\theta_y - M_{sv}\delta\theta_{y,y} - M_w\delta\theta_{y,yy} + m_y\delta\theta_y\right)\right] dy + \cdots = 0 \qquad \text{(A.8)}$$

or

$$\int_0^b \left[M\left(\frac{\delta\theta_y}{R} - \delta\theta_{z,y}\right) + Q\left(-\delta u_{,y} - \delta\theta_z\right) + M_{sv}\left(-\delta\theta_{y,y} - \frac{\delta\theta_z}{R}\right)\right.$$

$$\left. + M_w\left(-\delta\theta_{y,yy} - \frac{\delta\theta_{z,y}}{R}\right) + q_x\delta u + m_y\delta\theta_y\right] dy + \cdots = 0 \qquad \text{(A.9)}$$

or

$$\int_0^b \left[M\delta\left(\frac{\theta_y}{R} - \theta_{z,y}\right) + Q\delta\left(-u_{,y} - \theta_z\right) + M_{sv}\delta\left(-\theta_{y,y} - \frac{\theta_z}{R}\right)\right.$$

$$\left. + M_w\delta\left(-\theta_{y,yy} - \frac{\theta_{z,y}}{R}\right) + q_x\delta u + m_y\delta\theta_y\right] dy + \cdots = 0. \qquad \text{(A.10)}$$

Admitting

$$\theta_z = -u_{,y} \qquad \text{(A.11)}$$

so

$$-\int_0^b \left[M\delta\left(\frac{\theta_y}{R} - u_{,yy}\right) + M_{sv}\delta\left(\theta_{y,y} - \frac{u_{,y}}{R}\right)\right.$$

$$\left. + M_w\delta\left(\theta_{y,yy} - \frac{u_{,yy}}{R}\right) + q_x\delta u + m_y\delta\theta_y\right] dy + \cdots = 0 \qquad \text{(A.12)}$$

and

$$\kappa = -u_{,yy} - \frac{\theta_y}{R} \qquad \gamma_t = \theta_{y,y} - \frac{u_{,y}}{R} \qquad \kappa_w = \theta_{y,yy} - \frac{u_{,yy}}{R} \qquad \text{(A.13)}$$

Substituition of the constitutive equations

$$M = -EI\left(u_{,yy} + \frac{\theta_y}{R}\right) \qquad M_{sv} = GJ\left(\theta_{y,y} - \frac{u_{,y}}{R}\right) \qquad M_w = E\Gamma\left(\theta_{y,yy} - \frac{u_{,yy}}{R}\right) \qquad \text{(A.14)}$$

into the last equilibrium equation (A.5) leads to

$$m_y = -\frac{E\Gamma}{R}u_{,yyyy} + \frac{EI + GJ}{R}u_{,yy} + E\Gamma\theta_{y,yyyy} - GJ\theta_{y,yy} + \frac{EI}{R^2}\theta_y. \qquad \text{(A.15)}$$

This equation is equivalent to $\bar{M}$ in (3.4) as $\theta_y = w_{,x}$, by the geometry of the problem.

Appendix B – Displacement Coefficients and Matrix K

The coefficients u_{is}, $\bar{u}_{is}$, u_{ia}, $\bar{u}_{ia}$, v_{is}, $\bar{v}_{is}$, v_{ia}, $\bar{v}_{ia}$ and entries of matrix $\mathbf{K}$ are listed in the sequel.

Case I

Symmetric modes:

$$u_{is} = \gamma_i \frac{k^2 - \nu\gamma_i^2}{(k^2 + \gamma_i^2)^2} \qquad v_{is} = k\frac{k^2 + (2+\nu)\gamma_i^2}{(k^2 + \gamma_i^2)^2} \qquad i = 1, 2, 3, 4 \qquad \text{(B. 1)}$$

$$k_{1i} = u_{is}\gamma_i \cos\gamma_i\xi_0 \qquad k_{2i} = v_{is}\cos\gamma_i\xi_0 \qquad k_{3i} = \cos\gamma_i\xi_0$$

$$k_{4i} = -\frac{D}{R}\gamma_i^2 \cos\gamma_i\xi_0 - A\gamma_i \sin\gamma_i\xi_0 + Bu_{is}\sin\gamma_i\xi_0 \qquad i = 1, 2, 3, 4 \qquad \text{(B. 2)}$$

in which

$$A = \sqrt{\frac{p_{cl}}{2Eh}}\left(\frac{E_f I_f}{R^2} + \frac{n^2\pi^2}{b^2}G_f J_f + \frac{n^4\pi^4}{b^4}E_f\Gamma_f\right)$$

$$B = \frac{n^2\pi^2}{b^2}\left(\frac{p_{cl}}{2Eh}\right)^{3/2}\left(E_f I_f + G_f J_f + \frac{n^2\pi^2}{b^2}E_f\Gamma_f\right). \qquad \text{(B. 3)}$$

Antisymmetric modes:

$$(u_{ia}, v_{ia}) = (-u_{is}, v_{is}) \qquad i = 1, 2, 3, 4 \qquad \text{(B. 4)}$$

$$k_{1i} = -u_{ia}\gamma_i \sin\gamma_i\xi_0 \qquad k_{2i} = v_{ia}\sin\gamma_i\xi_0 \qquad k_{3i} = \sin\gamma_i\xi_0$$

$$k_{4i} = -\frac{D}{R}\gamma_i^2 \sin\gamma_i\xi_0 + A\gamma_i\cos\gamma_i\xi_0 + Bu_{ia}\cos\gamma_i\xi_0 \qquad i = 1, 2, 3, 4. \qquad \text{(B. 5)}$$

Case II

Symmetric modes:

$$(u_{1s}, u_{3s}, u_{4s}, v_{1s}, v_{3s}, v_{4s}) = (u_{1s}, u_{3s}, u_{4s}, v_{1s}, v_{3s}, v_{4s}) \text{ of Case I}$$

$$u_{2s} = \frac{1+\nu}{4k^2} \qquad v_{2s} = \frac{1}{2k^2}$$

$$(\bar{u}_{2s}, \bar{v}_{2s}) = (-u_{1s}, v_{1s}) \tag{B.6}$$

$$k_{1i} = u_{is}\gamma_i \cos\gamma_i\xi_0 \qquad k_{12} = u_{2s}\gamma_1 \cos\gamma_1\xi_0 + \bar{u}_{2s}(\cos\gamma_1\xi_0 - \gamma_1\xi_0 \sin\gamma_1\xi_0)$$

$$k_{2i} = v_{is} \cos\gamma_i\xi_0 \qquad k_{22} = v_{2s} \cos\gamma_1\xi_0 + \bar{v}_{2s}\xi_0 \sin\gamma_1\xi_0$$

$$k_{3i} = \cos\gamma_i\xi_0 \qquad k_{32} = \xi_0 \sin\gamma_1\xi_0$$

$$k_{4i} = -\frac{D}{R}\gamma_i^2 \cos\gamma_i\xi_0 - A\gamma_i \sin\gamma_i\xi_0 + Bu_{is} \sin\gamma_i\xi_0$$

$$k_{42} = \frac{D}{R}\gamma_1(2\cos\gamma_1\xi_0 - \gamma_1\xi_0 \sin\gamma_1\xi_0) + A(\sin\gamma_1\xi_0 + \gamma_1\xi_0 \cos\gamma_1\xi_0)$$
$$+B(u_{2s} \sin\gamma_1\xi_0 + \bar{u}_{2s}\xi_0 \cos\gamma_1\xi_0) \qquad i = 1, 3, 4 \tag{B.7}$$

Antisymmetric modes:

$$(u_{1a}, u_{3a}, u_{4a}, v_{1a}, v_{3a}, v_{4a}) = (u_{1a}, u_{3a}, u_{4a}, v_{1a}, v_{3a}, v_{4a}) \text{ of Case I}$$

$$(u_{2a}, \bar{u}_{2a}, v_{2a}, \bar{v}_{2a}) = (u_{2s}, -\bar{u}_{2s}, -v_{2s}, \bar{v}_{2s}) \tag{B.8}$$

$$k_{1i} = -u_{ia}\gamma_i \sin\gamma_i\xi_0 \qquad k_{12} = -u_{2a}\gamma_1 \sin\gamma_1\xi_0 + \bar{u}_{2a}(\sin\gamma_1\xi_0 + \gamma_1\xi_0 \cos\gamma_1\xi_0)$$

$$k_{2i} = v_{ia} \sin\gamma_i\xi_0 \qquad k_{22} = v_{2a} \sin\gamma_1\xi_0 + \bar{v}_{2a}\xi_0 \cos\gamma_1\xi_0$$

$$k_{3i} = \sin\gamma_i\xi_0 \qquad k_{32} = \xi_0 \cos\gamma_1\xi_0$$

$$k_{4i} = -\frac{D}{R}\gamma_i^2 \sin\gamma_i\xi_0 + A\gamma_i \cos\gamma_i\xi_0 + Bu_{ia} \cos\gamma_i\xi_0$$

$$k_{42} = -\frac{D}{R}\gamma_1(2\sin\gamma_1\xi_0 + \gamma_1\xi_0\cos\gamma_1\xi_0) + A(\cos\gamma_1\xi_0 - \gamma_1\xi_0\sin\gamma_1\xi_0)$$
$$+B(u_{2a}\cos\gamma_1\xi_0 + \bar{u}_{2a}\xi_0\sin\gamma_1\xi_0) \qquad i = 1, 3, 4 \qquad (\text{B}.9)$$

Case III

Symmetric modes:

$$(u_{3s}, u_{4s}, v_{3s}, v_{4s}) = (u_{3s}, u_{4s}, v_{3s}, v_{4s}) \text{ of Case I}$$

$$u_{1s} = \frac{\nu - 1}{4\beta_1} \qquad \bar{u}_{1s} = \frac{(1+\nu)\alpha_1}{4\beta_1^2}$$

$$(u_{2s}, \bar{u}_{2s}) = (\bar{u}_{1s}, -u_{1s})$$

$$v_{1s} = \frac{1}{2k} + \frac{(1+\nu)k}{4\beta_1^2} \qquad \bar{v}_{1s} = \frac{\alpha_1}{2k\beta_1}$$

$$(v_{2s}, \bar{v}_{2s}) = (-\bar{v}_{1s}, v_{1s}) \qquad (\text{B}.10)$$

$$k_{1i} = (\alpha_1 u_{is} + \beta_1\bar{u}_{is})\cosh\alpha_1\xi_0\cos\beta_1\xi_0 + (\alpha_1\bar{u}_{is} - \beta_1 u_{is})\sinh\alpha_1\xi_0\sin\beta_1\xi_0 \qquad i = 1, 2$$

$$k_{1i} = u_{is}\gamma_i\cos\gamma_i\xi_0 \qquad i = 3, 4$$

$$k_{2i} = v_{is}\sinh\alpha_1\xi_0\sin\beta_1\xi_0 + \bar{v}_{is}\cosh\alpha_1\xi_0\cos\beta_1\xi_0 \qquad i = 1, 2$$

$$k_{2i} = v_{is}\cos\gamma_i\xi_0 \qquad i = 3, 4$$

$$k_{31} = \sinh\alpha_1\xi_0\sin\beta_1\xi_0 \qquad k_{32} = \cosh\alpha_1\xi_0\cos\beta_1\xi_0 \qquad k_{3i} = \cos\gamma_i\xi_0 \qquad i = 3, 4$$

$$k_{41} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\sinh\alpha_1\xi_0\sin\beta_1\xi_0 + 2\alpha_1\beta_1\cosh\alpha_1\xi_0\cos\beta_1\xi_0]$$
$$+(A\alpha_1 + B\bar{u}_{1s})\cosh\alpha_1\xi_0\sin\beta_1\xi_0 + (A\beta_1 + Bu_{1s})\sinh\alpha_1\xi_0\cos\beta_1\xi_0$$

$$k_{42} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\cosh\alpha_1\xi_0\cos\beta_1\xi_0 - 2\alpha_1\beta_1\sinh\alpha_1\xi_0\sin\beta_1\xi_0]$$
$$+(A\alpha_1 + Bu_{2s})\sinh\alpha_1\xi_0\cos\beta_1\xi_0 - (A\beta_1 - B\bar{u}_{2s})\cosh\alpha_1\xi_0\sin\beta_1\xi_0$$

$$k_{4i} = -\frac{D}{R}\gamma_i^2 \cos\gamma_i\xi_0 - A\gamma_i \sin\gamma_i\xi_0 + Bu_{is}\sin\gamma_i\xi_0 \qquad i = 3,4 \tag{B.11}$$

Antisymmetric modes:

$$(u_{1a}, \bar{u}_{1a}, u_{2a}, \bar{u}_{2a}, v_{1a}, \bar{v}_{1a}, v_{2a}, \bar{v}_{2a}) = (-u_{1s}, \bar{u}_{1s}, \bar{u}_{1s}, u_{1s}, v_{1s}, -\bar{v}_{1s}, \bar{v}_{1s}, v_{1s})$$

$$(u_{3a}, u_{4a}, v_{3a}, v_{4a}) = (u_{3a}, u_{4a}, v_{3a}, v_{4a}) \text{ of Case I} \tag{B.12}$$

$$k_{1i} = (\alpha_1 u_{ia} - \beta_1 \bar{u}_{ia})\cosh\alpha_1\xi_0 \sin\beta_1\xi_0 + (\alpha_1\bar{u}_{ia} + \beta_1 u_{ia})\sinh\alpha_1\xi_0\cos\beta_1\xi_0 \qquad i = 1,2$$

$$k_{1i} = -u_{ia}\gamma_i \sin\gamma_i\xi_0 \qquad i = 3,4$$

$$k_{2i} = v_{ia}\sinh\alpha_1\xi_0\cos\beta_1\xi_0 + \bar{v}_{ia}\cosh\alpha_1\xi_0\sin\beta_1\xi_0 \qquad i = 1,2$$

$$k_{2i} = v_{ia}\sin\gamma_i\xi_0 \qquad i = 3,4$$

$$k_{31} = \sinh\alpha_1\xi_0\cos\beta_1\xi_0 \qquad k_{32} = \cosh\alpha_1\xi_0\sin\beta_1\xi_0 \qquad k_{3i} = \sin\gamma_i\xi_0 \qquad i = 3,4$$

$$k_{41} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\sinh\alpha_1\xi_0\cos\beta_1\xi_0 - 2\alpha_1\beta_1\cosh\alpha_1\xi_0\sin\beta_1\xi_0]$$
$$+(A\alpha_1 + B\bar{u}_{1a})\cosh\alpha_1\xi_0\cos\beta_1\xi_0 - (A\beta_1 - Bu_{1a})\sinh\alpha_1\xi_0\sin\beta_1\xi_0$$

$$k_{42} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\cosh\alpha_1\xi_0\sin\beta_1\xi_0 + 2\alpha_1\beta_1\sinh\alpha_1\xi_0\cos\beta_1\xi_0]$$
$$+(A\alpha_1 + Bu_{2a})\sinh\alpha_1\xi_0\sin\beta_1\xi_0 + (A\beta_1 + B\bar{u}_{2a})\cosh\alpha_1\xi_0\cos\beta_1\xi_0$$

$$k_{4i} = -\frac{D}{R}\gamma_i^2 \sin\gamma_i\xi_0 + A\gamma_i\cos\gamma_i\xi_0 + Bu_{ia}\cos\gamma_i\xi_0 \qquad i = 3,4 \tag{B.13}$$

Case IV

Symmetric modes:

$$(u_{1s}, \bar{u}_{1s}, u_{2s}, \bar{u}_{2s}, v_{1s}, \bar{v}_{1s}, v_{2s}, \bar{v}_{2s}) = (u_{1s}, \bar{u}_{1s}, u_{2s}, \bar{u}_{2s}, v_{1s}, \bar{v}_{1s}, v_{2s}, \bar{v}_{2s}) \text{ of Case III}$$

$$(u_{3s}, v_{3s}) = (u_{3s}, v_{3s}) \text{ of Case I} \tag{B.14}$$

The coefficients $(u_{4s},\ \bar{u}_{4s},\ v_{4s},\ \bar{v}_{4s})$ remain the same as those of $(u_{2s},\ \bar{u}_{2s},\ v_{2s},\ \bar{v}_{2s})$ found in Case II with γ_1 replaced by γ_3.

$$k_{1i} = (\alpha_1 u_{is} + \beta_1 \bar{u}_{is}) \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0 + (\alpha_1 \bar{u}_{is} - \beta_1 u_{is}) \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0 \qquad i = 1, 2$$

$$k_{13} = u_{3s} \gamma_3 \cos \gamma_3 \xi_0 \qquad k_{14} = u_{4s} \gamma_3 \cos \gamma_3 \xi_0 + \bar{u}_{4s} (\cos \gamma_3 \xi_0 - \gamma_3 \xi_0 \sin \gamma_3 \xi_0)$$

$$k_{2i} = v_{is} \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0 + \bar{v}_{is} \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0 \qquad i = 1, 2$$

$$k_{23} = v_{3s} \cos \gamma_3 \xi_0 \qquad k_{24} = v_{4s} \cos \gamma_3 \xi_0 + \bar{v}_{4s} \xi_0 \sin \gamma_3 \xi_0$$

$$k_{31} = \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0 \qquad k_{32} = \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0$$

$$k_{33} = \cos \gamma_3 \xi_0 \qquad k_{34} = \xi_0 \sin \gamma_3 \xi_0$$

$$k_{41} = \frac{D}{R} [(\alpha_1^2 - \beta_1^2) \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0 + 2\alpha_1 \beta_1 \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0]$$
$$+ (A\alpha_1 + B\bar{u}_{1s}) \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0 + (A\beta_1 + Bu_{1s}) \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0$$

$$k_{42} = \frac{D}{R} [(\alpha_1^2 - \beta_1^2) \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0 - 2\alpha_1 \beta_1 \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0]$$
$$+ (A\alpha_1 + Bu_{2s}) \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0 - (A\beta_1 - B\bar{u}_{2s}) \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0$$

$$k_{43} = -\frac{D}{R} \gamma_3^2 \cos \gamma_3 \xi_0 - A\gamma_3 \sin \gamma_3 \xi_0 + Bu_{3s} \sin \gamma_3 \xi_0$$

$$k_{44} = \frac{D}{R} \gamma_3 (2 \cos \gamma_3 \xi_0 - \gamma_3 \xi_0 \sin \gamma_3 \xi_0) + A(\sin \gamma_3 \xi_0 + \gamma_3 \xi_0 \cos \gamma_3 \xi_0)$$
$$+ B(u_{4s} \sin \gamma_3 \xi_0 + \bar{u}_{4s} \xi_0 \cos \gamma_3 \xi_0) \qquad \text{(B.15)}$$

Antisymmetric modes:

$$(u_{1a}, \bar{u}_{1a}, u_{2a}, \bar{u}_{2a}, v_{1a}, \bar{v}_{1a}, v_{2a}, \bar{v}_{2a}) = (u_{1a}, \bar{u}_{1a}, u_{2a}, \bar{u}_{2a}, v_{1a}, \bar{v}_{1a}, v_{2a}, \bar{v}_{2a}) \text{ of Case III}$$

$$(u_{3a}, v_{3a}) = (u_{3a}, v_{3a}) \text{ of Case I} \qquad \text{(B.16)}$$

The coefficients $(u_{4a},\ \bar{u}_{4a},\ v_{4a},\ \bar{v}_{4a})$ remain the same as those of $(u_{2a},\ \bar{u}_{2a},\ v_{2a},\ \bar{v}_{2a})$ found in Case II with γ_1 replaced by γ_3.

$$k_{1i} = (\alpha_1 u_{ia} - \beta_1 \bar{u}_{ia}) \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0 + (\alpha_1 \bar{u}_{ia} + \beta_1 u_{ia}) \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0 \qquad i = 1, 2$$

$$k_{13} = -u_{3a}\gamma_3 \sin\gamma_3\xi_0 \qquad k_{14} = -u_{4a}\gamma_3 \sin\gamma_3\xi_0 + \bar{u}_{4a}(\sin\gamma_3\xi_0 + \gamma_3\xi_0 \cos\gamma_3\xi_0)$$

$$k_{2i} = v_{ia} \sinh\alpha_1\xi_0 \cos\beta_1\xi_0 + \bar{v}_{ia} \cosh\alpha_1\xi_0 \sin\beta_1\xi_0 \qquad i = 1, 2$$

$$k_{23} = v_{3a} \sin\gamma_3\xi_0 \qquad k_{24} = v_{4a} \sin\gamma_3\xi_0 + \bar{v}_{4a}\xi_0 \cos\gamma_3\xi_0$$

$$k_{31} = \sinh\alpha_1\xi_0 \cos\beta_1\xi_0 \qquad k_{32} = \cosh\alpha_1\xi_0 \sin\beta_1\xi_0$$

$$k_{33} = \sin\gamma_3\xi_0 \qquad k_{34} = \xi_0 \cos\gamma_3\xi_0$$

$$k_{41} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\sinh\alpha_1\xi_0 \cos\beta_1\xi_0 - 2\alpha_1\beta_1 \cosh\alpha_1\xi_0 \sin\beta_1\xi_0]$$
$$+(A\alpha_1 + B\bar{u}_{1a})\cosh\alpha_1\xi_0 \cos\beta_1\xi_0 - (A\beta_1 - Bu_{1a})\sinh\alpha_1\xi_0 \sin\beta_1\xi_0$$

$$k_{42} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\cosh\alpha_1\xi_0 \sin\beta_1\xi_0 + 2\alpha_1\beta_1 \sinh\alpha_1\xi_0 \cos\beta_1\xi_0]$$
$$+(A\alpha_1 + Bu_{2a})\sinh\alpha_1\xi_0 \sin\beta_1\xi_0 + (A\beta_1 + B\bar{u}_{2a})\cosh\alpha_1\xi_0 \cos\beta_1\xi_0$$

$$k_{43} = -\frac{D}{R}\gamma_3^2 \sin\gamma_3\xi_0 + A\gamma_3 \cos\gamma_3\xi_0 + Bu_{3a} \cos\gamma_3\xi_0$$

$$k_{44} = -\frac{D}{R}\gamma_3(2\sin\gamma_3\xi_0 + \gamma_3\xi_0 \cos\gamma_3\xi_0) + A(\cos\gamma_3\xi_0 - \gamma_3\xi_0 \sin\gamma_3\xi_0)$$
$$+B(u_{4a} \cos\gamma_3\xi_0 + \bar{u}_{4a}\xi_0 \sin\gamma_3\xi_0) \qquad \text{(B.17)}$$

Case V

Symmetric modes:

$$(u_{1s}, \bar{u}_{1s}, u_{2s}, \bar{u}_{2s}, v_{1s}, \bar{v}_{1s}, v_{2s}, \bar{v}_{2s}) = (u_{1s}, \bar{u}_{1s}, u_{2s}, \bar{u}_{2s}, v_{1s}, \bar{v}_{1s}, v_{2s}, \bar{v}_{2s}) \text{ of Case III.} \qquad \text{(B.18)}$$

The coefficients $(u_{3s}, \bar{u}_{3s}, v_{3s}, \bar{v}_{3s})$ and $(u_{4s}, \bar{u}_{4s}, v_{4s}, \bar{v}_{4s})$ remain the same as those of $(u_{1s}, \bar{u}_{1s}, v_{1s}, \bar{v}_{1s})$ and $(u_{2s}, \bar{u}_{2s}, v_{2s}, \bar{v}_{2s})$, respectively, with α_1 and β_1 replaced by α_2 and β_2.

$$k_{1i} = (\alpha_1 u_{is} + \beta_1\bar{u}_{is})\cosh\alpha_1\xi_0 \cos\beta_1\xi_0 + (\alpha_1\bar{u}_{is} - \beta_1 u_{is})\sinh\alpha_1\xi_0 \sin\beta_1\xi_0 \qquad i = 1, 2$$

$$k_{1i} = (\alpha_2 u_{is} + \beta_2\bar{u}_{is})\cosh\alpha_2\xi_0 \cos\beta_2\xi_0 + (\alpha_2\bar{u}_{is} - \beta_2 u_{is})\sinh\alpha_2\xi_0 \sin\beta_2\xi_0 \qquad i = 3, 4$$

$$k_{2i} = v_{is} \sinh\alpha_1\xi_0 \sin\beta_1\xi_0 + \bar{v}_{is} \cosh\alpha_1\xi_0 \cos\beta_1\xi_0 \qquad i = 1, 2$$

$$k_{2i} = v_{is} \sinh \alpha_2 \xi_0 \sin \beta_2 \xi_0 + \bar{v}_{is} \cosh \alpha_2 \xi_0 \cos \beta_2 \xi_0 \qquad i = 3, 4$$

$$k_{31} = \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0 \qquad k_{32} = \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0$$

$$k_{33} = \sinh \alpha_2 \xi_0 \sin \beta_2 \xi_0 \qquad k_{34} = \cosh \alpha_2 \xi_0 \cos \beta_2 \xi_0$$

$$k_{41} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2) \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0 + 2\alpha_1 \beta_1 \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0] \\ +(A\alpha_1 + B\bar{u}_{1s}) \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0 + (A\beta_1 + Bu_{1s}) \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0$$

$$k_{42} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2) \cosh \alpha_1 \xi_0 \cos \beta_1 \xi_0 - 2\alpha_1 \beta_1 \sinh \alpha_1 \xi_0 \sin \beta_1 \xi_0] \\ +(A\alpha_1 + Bu_{2s}) \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0 - (A\beta_1 - B\bar{u}_{2s}) \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0$$

$$k_{43} = \frac{D}{R}[(\alpha_2^2 - \beta_2^2) \sinh \alpha_2 \xi_0 \sin \beta_2 \xi_0 + 2\alpha_2 \beta_2 \cosh \alpha_2 \xi_0 \cos \beta_2 \xi_0] \\ +(A\alpha_2 + B\bar{u}_{3s}) \cosh \alpha_2 \xi_0 \sin \beta_2 \xi_0 + (A\beta_2 + Bu_{3s}) \sinh \alpha_2 \xi_0 \cos \beta_2 \xi_0$$

$$k_{44} = \frac{D}{R}[(\alpha_2^2 - \beta_2^2) \cosh \alpha_2 \xi_0 \cos \beta_2 \xi_0 - 2\alpha_2 \beta_2 \sinh \alpha_2 \xi_0 \sin \beta_2 \xi_0] \\ +(A\alpha_2 + Bu_{4s}) \sinh \alpha_2 \xi_0 \cos \beta_2 \xi_0 - (A\beta_2 - B\bar{u}_{4s}) \cosh \alpha_2 \xi_0 \sin \beta_2 \xi_0 \qquad (\mathrm{B.19})$$

Antisymmetric modes:

$$(u_{1a}, \bar{u}_{1a}, u_{2a}, \bar{u}_{2a}, v_{1a}, \bar{v}_{1a}, v_{2a}, \bar{v}_{2a}) = (-u_{1s}, \bar{u}_{1s}, \bar{u}_{1s}, u_{1s}, v_{1s}, -\bar{v}_{1s}, \bar{v}_{1s}, v_{1s}) \qquad (\mathrm{B.20})$$

The coefficients $(u_{3a}, \bar{u}_{3a}, v_{3a}, \bar{v}_{3a})$ and $(u_{4a}, \bar{u}_{4a}, v_{4a}, \bar{v}_{4a})$ remain the same as those of $(u_{1a}, \bar{u}_{1a}, v_{1a}, \bar{v}_{1a})$ and $(u_{2a}, \bar{u}_{2a}, v_{2a}, \bar{v}_{2a})$, respectively, with α_1 and β_1 replaced by α_2 and β_2.

$$k_{1i} = (\alpha_1 u_{ia} - \beta_1 \bar{u}_{ia}) \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0 + (\alpha_1 \bar{u}_{ia} + \beta_1 u_{ia}) \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0 \qquad i = 1, 2$$

$$k_{1i} = (\alpha_2 u_{ia} - \beta_2 \bar{u}_{ia}) \cosh \alpha_2 \xi_0 \sin \beta_2 \xi_0 + (\alpha_2 \bar{u}_{ia} + \beta_2 u_{ia}) \sinh \alpha_2 \xi_0 \cos \beta_2 \xi_0 \qquad i = 3, 4$$

$$k_{2i} = v_{ia} \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0 + \bar{v}_{ia} \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0 \qquad i = 1, 2$$

$$k_{2i} = v_{ia} \sinh \alpha_2 \xi_0 \cos \beta_2 \xi_0 + \bar{v}_{ia} \cosh \alpha_2 \xi_0 \sin \beta_2 \xi_0 \qquad i = 3, 4$$

$$k_{31} = \sinh \alpha_1 \xi_0 \cos \beta_1 \xi_0 \qquad k_{32} = \cosh \alpha_1 \xi_0 \sin \beta_1 \xi_0$$

$$k_{33} = \sinh \alpha_2 \xi_0 \cos \beta_2 \xi_0 \qquad k_{34} = \cosh \alpha_2 \xi_0 \sin \beta_2 \xi_0$$

$$k_{41} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\sinh\alpha_1\xi_0\cos\beta_1\xi_0 - 2\alpha_1\beta_1\cosh\alpha_1\xi_0\sin\beta_1\xi_0]$$
$$+(A\alpha_1 + B\bar{u}_{1a})\cosh\alpha_1\xi_0\cos\beta_1\xi_0 - (A\beta_1 - Bu_{1a})\sinh\alpha_1\xi_0\sin\beta_1\xi_0$$

$$k_{42} = \frac{D}{R}[(\alpha_1^2 - \beta_1^2)\cosh\alpha_1\xi_0\sin\beta_1\xi_0 + 2\alpha_1\beta_1\sinh\alpha_1\xi_0\cos\beta_1\xi_0]$$
$$+(A\alpha_1 + Bu_{2a})\sinh\alpha_1\xi_0\sin\beta_1\xi_0 + (A\beta_1 + B\bar{u}_{2a})\cosh\alpha_1\xi_0\cos\beta_1\xi_0$$

$$k_{43} = \frac{D}{R}[(\alpha_2^2 - \beta_2^2)\sinh\alpha_2\xi_0\cos\beta_2\xi_0 - 2\alpha_2\beta_2\cosh\alpha_2\xi_0\sin\beta_2\xi_0]$$
$$+(A\alpha_2 + B\bar{u}_{3a})\cosh\alpha_2\xi_0\cos\beta_2\xi_0 - (A\beta_2 - Bu_{3a})\sinh\alpha_2\xi_0\sin\beta_2\xi_0$$

$$k_{44} = \frac{D}{R}[(\alpha_2^2 - \beta_2^2)\cosh\alpha_2\xi_0\sin\beta_2\xi_0 + 2\alpha_2\beta_2\sinh\alpha_2\xi_0\cos\beta_2\xi_0]$$
$$+(A\alpha_2 + Bu_{4a})\sinh\alpha_2\xi_0\sin\beta_2\xi_0 + (A\beta_2 + B\bar{u}_{4a})\cosh\alpha_2\xi_0\cos\beta_2\xi_0 \qquad (B.21)$$

FOLHA DE REGISTRO DO DOCUMENTO

1. CLASSIFICAÇÃO/TIPO	2. DATA	3. REGISTRO N°	4. N° DE PÁGINAS
TC	21 de novembro de 2017	DCTA/ITA/TC-090/2017	43f

5. TÍTULO E SUBTÍTULO:

Exact solution for buckling of axially-compressed cylindrical panels with frames attached to the circular edges.

6. AUTOR(ES):

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7. INSTITUIÇÃO(ÕES)/ÓRGÃO(S) INTERNO(S)/DIVISÃO(ÕES):

Instituto Tecnológico de Aeronáutica – ITA

8. PALAVRAS-CHAVE SUGERIDAS PELO AUTOR:

1. Estruturas. 2. Flambagem 2. 3. Paineis Cilíndricos.

9.PALAVRAS-CHAVE RESULTANTES DE INDEXAÇÃO:

Flambagem; Placas reforçadas; Corpos cilíndricos; Encurvamento; Engenharia estrutural.

10. APRESENTAÇÃO: **X Nacional** **Internacional**

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11. RESUMO:

Esse trabalho apresenta uma solução exata para o problema de valor de contorno que descreve a flambagem linear por compressão axial de painéis cilíndricos com enrijecedores acoplados às bordas curvas. As condições de contorno diferem do problema clássico simplesmente apoiado, em geral assumido para fins de projeto, no sentido de que a torsão resistida pelos enrijecedores são levadas em conta. A presença de dos enrijecedores torna os resultados obtidos aqui de grande interesse prático e valiosos como dados para referência.

12. GRAU DE SIGILO:

(X) OSTENSIVO **( ) RESERVADO** **( ) SECRETO**